# Asymptotic properties of Lévy flights in quenched random fields

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Long-time asymptotic behavior of the probability distribution function of Lévy flights in quenched random fields is analyzed with the use of field-theoretic renormalization group. This problem has been recently studied with the aid of a dynamic renormalization group based on the momentum-shell integration method [H.C. Fogedby, Phys. Rev. E **58**, 1690 (1998)]. While a great deal of the results of the quoted paper are confirmed by the present analysis, it is also shown that random field with long-range spatial correlations gives rise to asymptotic behavior with the dynamic critical exponent $z$ less than the step index $f$ of the Lévy flights, for a finite range of values of $f$ contrary to the conjecture that always $z=f$. In particular, in divergenceless random field $z=d/2+1-\alpha<f$, when $\alpha<1+d/2-f$ and $d<2+2\alpha$, and correlations fall off as $r^{-d+2\alpha}$. The physical content of a new critical dimension proposed in the aforementioned paper in connection with the anomalous scaling of the diffusion coefficient is also discussed.

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## I. INTRODUCTION

Recently, the momentum-shell version of the renormalization-group (RG) approach to the analysis of long-time large-scale asymptotic behavior of probability distribution functions was applied to Lévy flights in various quenched random fields [1,2]. Lévy flights are a generalization of random walks with a step distribution $p(l)$ falling off as a power of the step length $p(l)\propto l^{-1-f}$ with the step index $0<f<2$. The probability distribution function (PDF) $P(t,\mathbf{r})$ of the position $\mathbf{r}$ of a test particle in the external field $\mathbf{F}$ obeys the following Fokker-Planck equation:

$$\frac{\partial P}{\partial t}=-D_1(-\nabla^2)^{f/2}P+D_2\nabla^2P-\nabla[\mathbf{F}P]. \tag{1}$$

Here, the initial condition $P(0,\mathbf{r})=\delta(\mathbf{r})$ will be used. In Eq. (1) the fractional power of $\nabla^2$ is defined through the Fourier transform. The ordinary diffusion term is brought about by the small-scale part of the step distribution.

The zero-mean Gaussian distribution of the random field is determined by the correlation function

$$\langle F_m(\mathbf{r})F_n(\mathbf{r}')\rangle=C_{mn}(\mathbf{r}-\mathbf{r}'). \tag{2}$$

In the generic case the correlation function [3–5] consists of independent transverse and longitudinal parts $C=C^T+C^L$, where (in $d$-dimensional space)

$$C^T_{mn}(\mathbf{r})=g_T\int\frac{d\mathbf{k}}{(2\pi)^d}\frac{e^{i\mathbf{r}\cdot\mathbf{k}}}{k^{2\alpha}}\left(\delta_{mn}-\frac{k_mk_n}{k^2}\right),$$

$$C^L_{mn}(\mathbf{r})=g_L\int\frac{d\mathbf{k}}{(2\pi)^d}\frac{e^{i\mathbf{r}\cdot\mathbf{k}}}{k^{2\alpha}}\frac{k_mk_n}{k^2}, \tag{3}$$

where the coupling constants $g_T$ and $g_L$ measure the intensity of the correlations of solenoidal and potential parts, respectively, of the random field. The correlation function of the isotropic short-range case $C_{mn}(\mathbf{r})=g\delta_{mn}\delta(\mathbf{r})$ is recovered for $g_T=g_L=g$ and $\alpha=0$.

The motion of a test particle may be heavily affected by the random field leading to anomalous diffusion, i.e., to the long-time asymptotic behavior of the PDF in the form

$$P(t,\mathbf{r})=t^{-d/z}R(\mathbf{r}t^{-1/z}), \tag{4}$$

with the dynamic critical exponent $z\neq 2$. In Eq. (4) $R$ is a dimensionless scaling function.

A perturbative solution of the stochastic problem (1)–(3) becomes inconsistent below a critical dimension $d_c$, at and below which contributions from the small-wave-number (IR) region in the Fourier-transformed problem give rise to effective coupling constants growing with time. The critical dimension is determined from the condition that the coupling constants $g_T$ and $g_L$ are dimensionless. For instance, $d_c=2f-2$ for Lévy flights with the step index $f$ in a quenched field with isotropic local correlations.

These IR divergences may be dealt with by the use of the renormalization group in the critical dimension, where they can be transferred to the large-momentum region. The results may be extented below the critical dimension in the form of a $d_c-d$ expansion [6]. Above the critical dimension the perturbative solution is consistent and the leading asymptotic part of the PDF in the random field is the same as in zero field. When $d<d_c$, the higher order contributions to the PDF affect its structure and the resulting limiting distribution is not stable.

In Ref. [1] the asymptotic analysis was carried out by the use of the momentum-shell integration method for a generic random field with independent solenoidal and potential parts and both short-range and long-range correlations. Among the basic results of Ref. [1] was the observation that fluctuations of the drift field give rise to effective diffusive contributions nonanalytic in $k^2$, which was interpreted as another critical dimension (higher than that defined above by dimensional

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arguments) appearing for the model of Lévy flights in quenched random fields. It will be shown in this paper that these nonanalytic contributions remain subleading and do not spoil the ordinary perturbation expansion until the usual critical dimension $d_c$ is reached, whereas for $d \leqslant d_c$ the drift-field fluctuations generate leading terms (this effect may be described as coupling constants growing with time), which calls for a renormalization of the perturbation expansion.

It was also conjectured in Ref. [1] that the dynamic critical exponent $z=f$ irrespective of the dimension of space and the properties of the quenched disorder. This is in indeed so in large regions of the parameter space spanned by $d$ and $f$, but it will be shown below that there are also regions in which $z<f$ due to the transport-enhancing effect of the solenoidal part of the drift field with long-range correlations.

This paper is organized as follows. In Sec. II subtleties in the calculation of small-wave-number divergences are analyzed and the role of the usual critical dimension and that put forward in Ref. [1] discussed. Interplay of the rare long jumps of Lévy flights and solenoidal random advection in the generation of anomalous diffusion is analyzed in Sec. III with the aid of the field-theoretic RG. Section IV is devoted to conclusions.

## II. DIVERGENCES IN THE PERTURBATION THEORY AT LARGE SCALES

I will use the field-theoretic setup to formulate the perturbative solution of the stochastic problem (1), (2). The PDF determined by Eq. (1) and averaged over the random force may be written in the form of a functional integral (see, e.g., Ref. [6])

$$P(t,\mathbf{r})=\int \mathcal{D}\phi\mathcal{D}\tilde{\phi}\mathcal{D}\mathbf{F}\,\phi(t,\mathbf{r})\tilde{\phi}(0,\mathbf{0})e^{S(\phi,\tilde{\phi},\mathbf{F})},$$

where the "action" is of the form

$$S(\phi,\tilde{\phi},\mathbf{F})=\int d\mathbf{r}\,dt\,\tilde{\phi}[-\partial_t-D_1(-\nabla^2)^{f/2}+D_2\nabla^2]\phi - \frac{1}{2}\int d\mathbf{r}\,d\mathbf{r}'\sum_{m,n}F_m(\mathbf{r})C^{-1}_{mn}(\mathbf{r}-\mathbf{r}')F_n(\mathbf{r}') + \int d\mathbf{r}\,dt\,\mathbf{F}\phi\nabla\tilde{\phi}. \tag{5}$$

The standard renormalization theorem [7] is constructed for interactions local in space. For the short-range isotropic correlation function $C_{mn}(\mathbf{r}-\mathbf{r}')=g\delta_{mn}\delta(\mathbf{r}-\mathbf{r}')$ integration over the Gaussian drift field yields a local in space effective action for the fields $\phi$ and $\tilde{\phi}$. However, in the generic case a nonlocal interaction term results, and then it is thus preferable not to integrate out the drift field $\mathbf{F}$ in the action (5).

To remind the origin of the divergence problems I calculate the simplest fluctuation correction to the zero-field PDF

$$P_0(\omega,\mathbf{k})=\frac{1}{-i\omega+D_1k^f+D_2k^2}$$

of Eq. (1). It is customary to express the fluctuation corrections in terms of a self-energy function $\Sigma_{\phi\tilde{\phi}}(\omega,\mathbf{k})$ defined through the full PDF $P(\omega,\mathbf{k})$ (i.e., averaged over the drift field $\mathbf{F}$) as

$$P(\omega,\mathbf{k})=\frac{1}{-i\omega+D_1k^f+D_2k^2-\Sigma_{\phi\tilde{\phi}}(\omega,\mathbf{k})}. \tag{6}$$

In order to make wave-number integrals in the perturbative expansion of the self-energy finite it is necessary to introduce a large-wave-number (UV) regularization. This is often done by the use of a cutoff parameter $\Lambda$ for the wave numbers. Physically, $\Lambda$ is the inverse of the microscopic physical length scale, below which the macroscopic description of Eqs. (1), (2) does not apply.

The actual renormalization is carried out in the critical dimension, in which the UV and possible IR divergences are related. The standard machinery of the renormalization theory allows to deal consistently with the UV divergences only, therefore it is necessary to distinguish contributions to singularities coming from the UV and IR regions of the wave-vector space. To this end, it is useful to introduce also a small-wave-number regularization. There are many ways to carry out both regularizations, but some care should be taken in doing this, e.g., to preserve important symmetries of the model.

The IR regularization may be introduced, for instance, by adding a decay term $-m^2P$ to the right-hand side of the original Fokker-Planck equation (1). This is equivalent to an imaginary shift in the frequency $\omega\rightarrow\omega+im^2$. In this case there is no need for a separate small-wave-number regularization in the quenched model, since the frequency acts as an effective IR-cutoff parameter due to the absence of integrations over frequencies.

In Ref. [1] the short-range isotropic correlation function $C_{mn}(\mathbf{k})=g\delta_{mn}$ was used for illustration of the calculation of divergent contributions to the self-energy. This particular example is, however, somewhat misleading, since the presence of the divergence conjectured in Ref. [1] in the wave-vector expansion of the one-loop contribution to the PDF:

$$\Sigma^{(1)}_{\phi\tilde{\phi}}(\omega,\mathbf{k})=-g\int\frac{d\mathbf{q}}{(2\pi)^d}\times\sum_l\frac{k_l(k_l+q_l)T(\mathbf{k},\mathbf{q},m,\Lambda)}{[-i\omega+D_1(\mathbf{k}+\mathbf{q})^f+D_2(\mathbf{k}+\mathbf{q})^2]} \tag{7}$$

heavily depends on the choice of the regularization. In Eq. (7) $T$ is a, so far unspecified, function for the regularization of the wave-number integral.

The simplest way to introduce a regularization is to restrict all the integration variables to finite shells $m\leqslant|\mathbf{q}|\leqslant\Lambda$ in the wave-number integrals, which in Eq. (7) would correspond to the choice $T(\mathbf{k},\mathbf{q},m,\Lambda)=T(q)=\theta(q-m)-\theta(\Lambda-q)$, where $\theta$ is the Heaviside step function. This method has the disadvantage that the translational invariance is lost.

Translationally invariant regularization may be achieved by the use of the same function $T$ to restrict wave vectors flowing in the zero-field PDFs or drift-field correlation functions $P_0(\omega,\mathbf{k})\rightarrow P_0(\omega,\mathbf{k})T(k)$ or $C_{ij}(\mathbf{k})\rightarrow C_{ij}(\mathbf{k})T(k)$, respectively. For the one-loop graph (7) the latter choice results in the same integral as the simple cutoff of all integration variables. Regularization by a cutoff of the wave vector flowing in the correlation function, however, has the disadvantage that the regularized correlation funcion ceases to be local in space. Therefore, to preserve both translational invariance and local character of correlations the regularization procedure should be applied to the free-field PDF only.

The cutoff of integration variables has been used for regularization in Ref. [1], as can be seen from its Appendix B. This cutoff indeed leads to renormalization of the diffusion coefficient $D_2$ by a quantity divergent in the limit $m\to 0$, when the integral (7) is calculated at $\omega=0$.

However, if an IR regularization by the decay term is assumed, then the leading nonvanishing coefficient of the Taylor expansion of the function (7) with respect to $\mathbf{k}$ is given by the integral

$$I_{mn}(\mathbf{k})=-g\int_{q\leq\Lambda}\frac{d\mathbf{q}}{(2\pi)^d}\frac{\partial}{\partial k_m}\times\frac{(k_n+q_n)}{[m^2+D_1(\mathbf{k}+\mathbf{q})^f+D_2(\mathbf{k}+\mathbf{q})^2]}. \tag{8}$$

The derivatives with respect to $k_n$ act on a function of $\mathbf{k}+\mathbf{q}$ and may thus be replaced by the derivatives with respect to $q_n$. By virtue of the Gauss theorem, the integral (8) may be written as a surface integral over the surface of a sphere of radius $\Lambda$ in the wave-vector space:

$$I_{mn}(\mathbf{0})=-g\int\frac{dS}{(2\pi)^d}\frac{e_m e_n\Lambda}{[m^2+D_1\Lambda^f+D_2\Lambda^2]}=-g\frac{2\Lambda^d\pi^{d/2}\delta_{mn}}{\Gamma(d/2)d(2\pi)^d[m^2+D_1\Lambda^f+D_2\Lambda^2]},$$

where the unit vector $\mathbf{e}=\mathbf{q}/q$ has been introduced. Thus there is no divergence in the leading term of the wave-vector expansion of $\Sigma^{(1)}_{\phi\tilde{\phi}}$, when $m^2\to 0$:

$$\Sigma^{(1)}_{\phi\tilde{\phi}}(0,\mathbf{k})=-g\frac{2\Lambda^d k^2}{\Gamma(d/2)d(4\pi)^{d/2}[D_1\Lambda^f+D_2\Lambda^2]}+o(k^2).$$

Even a more radical result is obtained, if, in order to be sure that all the symmetries of the original model are preserved in the renormalized one, the regularization through the free-field PDF $[P_0(\omega,\mathbf{k})\to P_0(\omega,\mathbf{k})T(k)]$ is chosen. Then a simple change of variables shows that the integral in Eq. (7) is independent of the external wave vector $\mathbf{k}$ and thus due to rotational invariance vanishes identically.

This dependence on the choice of the regularization procedure reflects not only the subtleties in the choice of a consistent method of renormalization, but also the singular dependence of the full PDF on the frequency and wave number. Problems related to a consistent choice of a renormalization prescription become rather involved in higher orders due to overlapping of divergences in manyfold wave-vector integrals. Thus, care must be taken when inferring asymptotic properties of a model from its divergent perturbation expansion. I would like to emphasize that to date the only consistent regular method to analyze asymptotic behavior of a perturbative expansion is the field-theoretic renormalization group.

In general perturbative contributions to the self-energy function $\Sigma_{\phi\tilde{\phi}}(\omega,\mathbf{k})$ (6) are singular functions of $\omega$ and $\mathbf{k}$. This can be seen already at the one-loop level for unconstrained drift field. Consider, for instance, the one-loop integral with the transverse correlation function (I will use a regularization through the correlation function here for simplicity, because the transverse correlation function is nonlocal in space anyway)

$$\Sigma^{(1)T}_{\phi\tilde{\phi}}(\omega,\mathbf{k})=-\frac{g_T}{D_1}\int_{q\leq\Lambda}\frac{d\mathbf{q}}{(2\pi)^d}\times\sum_{m,n}\frac{k_m(k_n+q_n)}{[-i\omega/D_1+(\mathbf{k}+\mathbf{q})^f]}\left(\delta_{mn}-\frac{q_m q_n}{q^2}\right),$$

where I have retained only the leading terms in the limit of small frequency and wave number of the zero-field PDF.

Proceeding in the standard fashion [6] I investigate the scaling limit $\omega\to s^f\omega$ $k\to sk$ of the full PDF. After a suitable change of variables and taking into account that the external wave vectors factorize, I arrive at the expression

$$\Sigma^{(1)T}_{\phi\tilde{\phi}}(s^f\omega,s\mathbf{k})=-\frac{g_T}{D_1}\sum_{m,n}sk_m sk_n\int_{q\leq\Lambda/s}\frac{d\mathbf{q}}{(2\pi)^d}\times\frac{s^{d-f}}{[-i\omega/D_1+(\mathbf{k}+\mathbf{q})^f]}\left(\delta_{mn}-\frac{q_m q_n}{q^2}\right). \tag{9}$$

In the limit $s\to 0$ the effective UV cutoff $\Lambda/s\to\infty$ and the integral in Eq. (9) diverges as $s^{f-d}$, when $d>f$ (logarithmically, when $d=f$). Due to the prefactor, $s^{d+2-f}$, the whole expression $\Sigma^{(1)T}_{\phi\tilde{\phi}}(s^f\omega,s\mathbf{k})$ behaves as $s^2$ ($s^2\ln s$ in $d=f$) in the limit $s\to 0$, which is small compared with the zeroth-order term $P_0^{-1}(s^f\omega,s\mathbf{k})\sim s^f(-i\omega+D_1k^f)$.

Further, when $d<f$, the integral in Eq. (9) converges in the region of large wave numbers and its scaling behavior is given by $s^{d-f}$. This is a signal of singular behavior of the integral in the region of small wave numbers in the limit $s\to 0$. The borderline value $d=f$ is the first critical dimension put forward in Ref. [1]. However, to judge about the behavior of the whole perturbative correction $\Sigma^{(1)T}_{\phi\tilde{\phi}}$ in this limit, the scaling of the factorized wave vectors must be taken into account. Thus, for $d<f$ the one-loop term scales as $\Sigma^{(1)T}_{\phi\tilde{\phi}}(s^f\omega,s\mathbf{k})\sim s^{d+2-f}=s^{f+(d-2f+2)}$ being still subleading, while $d>2f-2$. Only for $d\leq 2f-2$ the perturbative correction scales in such a way that it becomes the leading contribution at this order. This may be expressed as $g_T s^{d-2f+2}$ becoming the effective expansion parameter and thus invalidating the perturbation expansion. Therefore, the usual perturbation theory becomes inapplicable only at and below the standard critical dimension $d_c=2f-2$.

The singular behavior of the integral in Eq. (9) for $d<f$ means that $\Sigma^{(1)T}_{\phi\tilde{\phi}}(0,\mathbf{k})\sim k^{d+2-f}$, and since the power of $k$ here is less than two and noninteger, its contribution to

the diffusion coefficient calculated as $D_2^{(1)} = -[\partial \Sigma_{\phi\tilde{\phi}}^{(1)T}(0,\mathbf{k})/\partial k^2]|_{k=0}$ does not exist. But there are also terms $\propto k^2$ in $\Sigma_{\phi\tilde{\phi}}^{(1)T}(0,\mathbf{k})$, and it would be more natural to regard them as the one-loop contribution to the effective counterpart of $D_2$ in the full PDF.

However, the major effect of the one-loop contribution $\Sigma_{\phi\tilde{\phi}}^{(1)T}(0,\mathbf{k})$ is that in the limit $\mathbf{k}\to 0$ it gives rise to a new nonanalytic term $\propto k^{d+2-f}$ subleading compared with the Lévy-flight term $\propto k^f$, but leading compared with the ordinary diffusion term $\propto k^2$. Analysis of higher-order contributions to the full PDF reveals that the $n$-loop contribution at zero frequency gives rise to a subleading term $\sim k^{d+2-f+(n-1)(d+2-2f)}$ when the power of $k$ is less than 2. Above the critical dimension $2f-2$ the scaling dimension of the coupling constant $g_T$ is negative, therefore the power of $k$ in these contributions increases with the order of perturbation expansion of the full PDF. When $d+2-f+(n-1)(d+2-2f)\geqslant 2$ only corrections to the ordinary diffusion term $\propto k^2$ are produced.

Thus, when the dimension of space is between the critical dimension and the step index: $2f-2<d<f$, new subleading nonanalytic terms in the PDF are brought about by the perturbative expansion. However, the leading asymptotic behavior of the PDF is not affected by these terms and the perturbation expansion itself remains perfectly consistent. Therefore, no renormalization of the expansion series is needed to find out the large-scale asymptotic properties of the model.

The appearance of these new nonanalytic terms means, however, that from the point of view of finding the large-scale asymptotics of the model, the ordinary diffusion term in Eq. (1) is excessive, unless the difference $2-f$ is small. In case of small $2-f$ it has to be taken into account that the renormalization of the ordinary diffusion term in the full PDF due to the fluctuations of the random field $\mathbf{F}$ gives rise to anomalous asymptotic behavior of the diffusion term which, depending on the properties of the random field, may even be the leading asymptotic contribution to the full PDF. This issue will be analyzed in detail in the following section.

## III. INTERPLAY OF LÉVY FLIGHTS AND SOLENOIDAL DRIFT

One important feature in the behavior of the Lévy flights in quenched random field has been overlooked practically completely in Ref. [1]. The rare long jumps of the test particle allow it to escape the traps created by the curl-free part of the generic random field. The divergenceless part of the drift, however, enhances the transport rate giving rise to superdiffusive behavior even in the case of ordinary random walks [3–5]. This feature is present also in the Lévy flights [8], and in certain regions of the parameter space leads to faster superdiffusive behavior than the rare long jumps of the Lévy flights.

Consider the case of long-range correlated random field, for which the renormalized action may be written in the form

$$S_R = \int d\mathbf{r}\,dt\,\tilde{\phi}[-\partial_t - D_{1R}(-\nabla^2)^{f/2} + Z_D D_{2R}\mu^{f-2}\nabla^2]\phi$$
$$-\frac{1}{2}\mu^{-(2f-2-d+2\alpha)}\int d\mathbf{r}\,d\mathbf{r}'\sum_{m,n} B_m(\mathbf{r})\times[C_R^T]^{-1}_{mn}(\mathbf{r}-\mathbf{r}')B_n(\mathbf{r}')$$
$$-\frac{1}{2}\mu^{-(2f-2-d+2\alpha)}\int d\mathbf{r}\,d\mathbf{r}'\sum_{m,n} E_m(\mathbf{r})\times[C_R^L]^{-1}_{mn}(\mathbf{r}-\mathbf{r}')E_n(\mathbf{r}') + Z_1\int d\mathbf{r}\,dt\,\mathbf{E}\phi\nabla\tilde{\phi}$$
$$+Z_2\int d\mathbf{r}\,dt\,\mathbf{B}\phi\nabla\tilde{\phi}, \qquad (10)$$

where the random drift field has been expressed as the sum of a solenoidal part $\mathbf{B}$ and a potential part $\mathbf{E}$: $\mathbf{F}=\mathbf{B}+\mathbf{E}$. In the action (10) the bare coupling constants $g_T$ and $g_L$ in the renormalized correlation function $C_R$ have been replaced by their renormalized counterparts $g_{TR}$, $g_{LR}$. The renormalization constants $Z_D$, $Z_1$, and $Z_2$ have been introduced to absorb the UV divergences of the model (their notation follows that of Ref. [5]). The wave-number scale-setting parameter is denoted by $\mu$. From a careful inspection of the structure of the fluctuation corrections [5] it follows that

$$Z_1 = Z_2$$

in the renormalized model. This relation implies that the ratio $\kappa = g_L/g_T$ is invariant under renormalization and it is sufficient to analyze the renormalization of the coupling constant $g_T$ only. The ratio $\kappa$ remains a free parameter of the model.

The standard critical dimension in this case is $d_c = 2f-2+2\alpha$. Note that in order to carry out the renormalization unambiguously one of the terms $D_{1R}\tilde{\phi}(-\nabla^2)^{f/2}\phi - Z_D D_{2R}\mu^{f-2}\tilde{\phi}\nabla^2\phi$ must be considered an interaction term [9]. In Eq. (10) the ordinary diffusion term is chosen as a part of the interaction, and the Lévy distribution term $\propto D_{1R}(-\nabla^2)^{f/2}$ is therefore not scaled under renormalization.

Thus, the connection between the renormalized and unrenormalized (bare) parameters is

$$D_{1R} = D_1,$$

$$Z_1^2 g_{TR}\mu^{2f-2-d+2\alpha} = g_T,$$

$$Z_D D_{2R}\mu^{f-2} = D_2.$$

The RG $\gamma$ and $\beta$ functions are defined as

$$\beta_v = \mu \frac{\partial}{\partial \mu}\bigg|_{D,g} v_R,$$

$$\beta_\zeta = \mu \frac{\partial}{\partial \mu}\bigg|_{D,g} \zeta_R,$$

$$\gamma_i = -\mu \frac{\partial}{\partial \mu}\bigg|_{D,g} \ln Z_i,$$

where the partial derivatives are calculated with fixed unrenormalized parameters, and $v_R = g_{TR}(D_1 + D_{2R})^{-2}$, $\zeta_R = D_{2R}/D_1$ are the dimensionless expansion parameters of the model.

The $\beta$ functions may be expressed through the renormalization-group functions of the problem of random walks in random environments [8]:

$$\beta_v = v_R[-(2f-2+2\alpha-d)+2\gamma_{1D}(v_R)-2\gamma_{DD}(v_R)] - \frac{2(2-f)v_R\zeta_R}{1+\zeta_R},$$

$$\beta_\zeta = \zeta_R(2-f)+(1+\zeta_R)\gamma_{DD}(v_R), \qquad (11)$$

which are labeled by an additional subscript $D$. At one-loop order the $\gamma$ functions in Eq. (11) are [5]

$$\gamma_{1D} = \frac{\kappa v_R}{2(1+\alpha)}, \qquad \gamma_{DD} = \frac{(\kappa-1-2\alpha)v_R}{2(1+\alpha)}.$$

The Callan-Symanzik equation for the renormalized PDF may be written as [5]

$$\left[ft\frac{\partial}{\partial t} - \mathbf{k}\frac{\partial}{\partial \mathbf{k}} + \beta_v \frac{\partial}{\partial v_R} + \beta_\zeta \frac{\partial}{\partial \zeta_R}\right] P_R(t,\mathbf{k}) = 0. \qquad (12)$$

Solution by the method of characteristics yields

$$P_R(s^{-f}t, s\mathbf{k}; v_R, \zeta_R) = P_R(t, \mathbf{k}; \overline{v}, \overline{\zeta}), \qquad (13)$$

where the $\overline{v}$ and $\overline{\zeta}$ are obtained from the equations

$$\int_{v_R}^{\overline{v}} \frac{dv}{\beta_v} = \ln s, \qquad \int_{\zeta_R}^{\overline{\zeta}} \frac{d\zeta}{\beta_\zeta} = \ln s. \qquad (14)$$

From Eqs. (13) and (14) it follows that in the limit $t \to \infty$ the PDF becomes scale invariant:

$$P_R(t,\mathbf{k}) \sim \tilde{R}(\mathbf{k}\, t^{1/f}), \qquad (15)$$

provided the renormalized coupling constants $v_R$ and $\zeta_R$ are in the basin of attraction of an IR-stable fixed point of Eq. (12), determined by the equations $\beta_v = \beta_\zeta = 0$ and the condition that the matrix of derivatives of the $\beta$ functions with respect to renormalized coupling constants is positive definite at the fixed point. Obviously $z = f$ in Eq. (15), as conjectured in Ref. [1].

Fixing now the scale of the $D_2 k^2$ term I arrive at the renormalized action

$$S = \int d\mathbf{r}\, dt\, \tilde{\phi}[-\partial_t - D_{1R}\mu^{2-f}(-\nabla^2)^{f/2} + Z_D D_{2R}\nabla^2]\phi - \frac{1}{2}\mu^{-(2+2\alpha-d)} \int d\mathbf{r}\, d\mathbf{r}' \sum_{m,n} B_m(\mathbf{r}) \times [C_R^T]^{-1}_{mn}(\mathbf{r}-\mathbf{r}')B_n(\mathbf{r}') - \frac{1}{2}\mu^{-(2+2\alpha-d)} \int d\mathbf{r}\, d\mathbf{r}' \sum_{m,n} E_m(\mathbf{r}) \times [C_R^L]^{-1}_{mn}(\mathbf{r}-\mathbf{r}')E_n(\mathbf{r}') + Z_1 \int d\mathbf{r}\, dt\, \mathbf{E}\phi\nabla\tilde{\phi} + Z_2 \int d\mathbf{r}\, dt\, \mathbf{B}\phi\nabla\tilde{\phi},$$

with the critical dimension $d_c = 2+2\alpha$ independent of the step index $f$. In this case the connection between renormalized and unrenormalized parameters is

$$D_{1R}\mu^{2-f} = D_1,$$

$$Z_1^2 g_{TR}\mu^{2+2\alpha-d} = g_T,$$

$$Z_D D_{2R} = D_2.$$

The Callan-Symanzik equation is now

$$\left[(2+\gamma_D)t\frac{\partial}{\partial t} - \mathbf{k}\frac{\partial}{\partial \mathbf{k}} + \beta_v \frac{\partial}{\partial v_R} + \beta_\chi \frac{\partial}{\partial \chi_R}\right] P_R(t,\mathbf{k}) = 0, \qquad (16)$$

where the dimensionless parameter $\chi = D_{1R}/D_{2R}$ and

$$\beta_\chi = \mu \frac{\partial}{\partial_\mu}\bigg|_{D,g} \chi_R.$$

The $\beta$ functions are slightly different from those of Eq. (11):

$$\beta_v = v_R[-(2+2\alpha-d)+2\gamma_{1D}(v_R)-2\gamma_{DD}(v_R)] - \frac{2(2-f)v_R\chi_R}{1+\chi_R},$$

$$\beta_\chi = \chi_R[-(2-f)-(1+\chi_R)\gamma_{DD}(v_R)].$$

As above, an analysis of the solution of Eq. (16) reveals that the coefficient of the differential operator $t\partial_t$, calculated at an IR-stable fixed point, yields the dynamic exponent $z = 2 + \gamma_D^*$, where $\gamma_D^*$ is the fixed-point value of the function $\gamma_D(v_R, \chi_R)$.

From the conditions of IR stability of the fixed points of Eqs. (12) and (16) it follows [8] that the dynamic exponent is less than the step index in the following cases.

(1) Divergenceless drift field: $C^L = 0$, $\alpha \geqslant 0$. In this case the dynamic exponent is calculated exactly due to absence of vertex renormalization

$$z = \frac{d}{2} + 1 - \alpha,$$

both for short-range ($\alpha=0$) and long-range ($\alpha>0$) correlated random field. Conditions of IR stability of the corresponding fixed point [8]

$$2-d+2\alpha>0, \quad 2f-d+2\alpha-2>0$$

imply that $z<f$ in this case. These expressions are valid for $\alpha\geqslant 0$, i.e., for both short-range and long-range correlated solenoidal random field.

(2) Unconstrained drift field with long-range correlations: $C^T\neq 0$, $C^L\neq 0$, $\alpha>0$. In this case the dynamic exponent has been calculated in the form of expansion in the small parameter $d_c-d=2+2\alpha-d$; at the leading nontrivial order

$$z=2+\frac{(\kappa-1-2\alpha)(2+2\alpha-d)}{2(1+2\alpha)}.$$

The stability conditions of the corresponding fixed point are

$$2-f+\frac{(\kappa-1-2\alpha)(2+2\alpha-d)}{2(1+2\alpha)}<0,$$

$$2+2\alpha-d>0, \quad \kappa<1+2\alpha,$$

and lead again to the conclusion that $z<f$. In other cases, when the dynamic exponent is calculable for the Lévy flights, the equality $z=f$ holds.

The possibility of fixing the scale of the ordinary diffusion term instead of the Lévy-flight term was in fact briefly considered also in Ref. [1] (Appendix E), but for a case where the diffusion-enhancing property of the divergenceless part of the drift field did not affect the asymptotic behavior. As shown above, superdiffusive behavior is in general due to both the rare long jumps of the Lévy flights and the advection by the solenoidal part of the drift field, and for a finite range of physically acceptable values of the parameters the dynamic exponent $z=2+\gamma_D^*<f$.

## IV. CONCLUSION

In this paper the renormalization of the Lévy flights in quenched random fields has been analyzed in view of the recent conjectures [1,2] about the long-time asymptotic properties of this model. Subtleties in the renormalization of the corresponding field-theoretic model related to the preservation of symmetries and locality properties have been illustrated in the example of the one-loop self-energy correction.

It has been shown that the large-scale long-time asymptotic behavior of the probability distribution function of the Lévy flights is affected by the appearance of new subleading nonanalytic terms in space dimensions $2f-2<d<f$, which are generated by fluctuations of the drift field together with the Lévy flight term in the zero-field PDF. However, the leading asymptotic behavior remains unaffected by this mechanism down to the critical dimension $d_c=2f-2$, at and below which the perturbation theory breaks down, and a renormalization-group treatment is called for. The ''critical dimension'' of Ref. [1] $d=f$ is thus related to the change in the analytic structure of the subleading terms of the PDF rather than to the divergence of the coefficient of the ordinary diffusion term in the full PDF.

It has been also shown that anomalous long-time asymptotic behavior may be also brought about by the renormalization of the ordinary diffusive term below the critical dimension of the ordinary diffusion $d_c=2+2\alpha$ (which is higher than that of the Lévy flights). In some cases this yields not only subleading but the leading contribution to the asymptotic behavior of the PDF. In particular, it has been shown that the superdiffusive asymptotic behavior of the Lévy flights is brought about by both the rare long jumps typical of the Lévy flights, and the diffusion-enhancing advection by the divergenceless part of the drift field. The leading asymptotic term has been shown to be due to the drift field advection for slowly enough falling off correlations of the drift field, and in this case the dynamic critical exponent $z<f$, contrary to the conjecture of Refs. [1,2].


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